Pergamon

Int. J. Heat Mass Transfer. Vol. 37, No. 6, pp. 1045–1048, 1994

0017–9310/94 $6.00+0.00

0017-9310(93)E0011-5

TECHNICAL NOTES

The optimal spacing of a stack of plates cooled by turbulent forced convection

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(*Received* 4 *August* 1993 *and in final form* 27 *October* 1993)

INTRODUCTION

A FUNDAMENTAL question in the design of finned heat exchanger surfaces and electronics packages is how to determine the spacing between heat generating plates in a stack of fixed volume. When the stack peak temperature (hot spot) is fixed, the optimal plate-to-plate spacing corresponds to the maximum heat transfer rate from the entire stack to the ambient fluid.

The optimal spacing problem was solved for configurations where the stack is cooled by natural convection [1–6]. Stacks cooled by forced convection were optimized only in cases where the flow is laminar [7–10]. The objective of this note is to report the optimal plate-to-plate spacing when the stack is cooled by turbulent forced convection.

ANALYSIS

With reference to Fig. 1, we seek to maximize the overall thermal conductance of the stack, $q'/(T_{max}-T_\infty)$, by selecting the plate-to-plate spacing D. The method of solution is analytical, and consists of (i) estimating the stack-ambient thermal conductance in the asymptotic regimes (small D, large D), and (ii) intersecting the two asymptotic solutions. The accuracy of this method relative to more exact methods was demonstrated in ref. [10], where the flow was laminar. This method is even more justified in the case of turbulent flow, because of the relatively greater uncertainty built into the turbulent heat and fluid flow correlations used for the asymptotic regimes.

In the following analysis, the coolant temperature (T_∞) and the representative order of magnitude of the plate temperature (T_{max}) are given. The pressure drop across the stack (ΔP) is fixed, as in applications where several stacks receive their coolant from the same plenum. The same analysis holds for configurations where the stack is immersed in a free stream U_0, because the effective pressure drop across the stack is then $\Delta P \cong (1/2)\rho U_0^2$, constant. The board thickness t is not necessarily negligible when compared with the board-to-board spacing D (see equation (6)). In other words, the number of boards in the stack of thickness H is $n = H/(D+t)$, where it is assumed that $n \gg 1$.

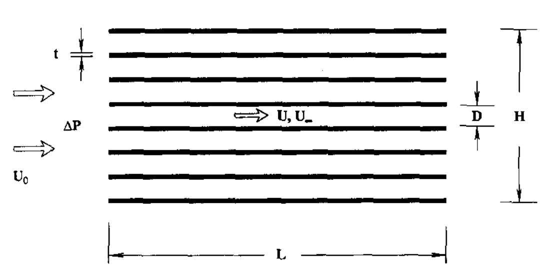

FIG. 1. Stack of equidistant heat generating plates cooled by turbulent forced convection.

Small D

When the board-to-board spacing is sufficiently small, the outlet temperature of the coolant is the same as the board temperature, and the total rate of heat transfer removed from the package is

$$q'_a = \dot{m}' c_P (T_{max} - T_\infty). \quad (1)$$

The mass flowrate is $\dot{m}' = n\rho UD$, where U is the mean velocity through each D channel with fully developed turbulent flow,

$$U = \left(\frac{D\Delta P}{\rho L f}\right)^{1/2}. \quad (2)$$

The friction factor f depends on the channel Reynolds number, as we will see in equation (11). In conclusion, if we combine equations (1) and (2) with $\dot{m}' = n\rho UD$ we obtain the $D \to 0$ asymptote of the overall thermal conductance:

$$\left(\frac{q'}{T_{max}-T_\infty}\right)_{D\to 0} = \frac{c_P H}{1+\frac{t}{D}}\left(\frac{\rho D \Delta P}{fL}\right)^{1/2}. \quad (3)$$

Large D

In the opposite extreme each board is lined by boundary layers, while the core of the channel of spacing D is filled by coolant of temperature T_∞ and core (outside the boundary layers) velocity U_∞. The latter is dictated by the force balance on the entire stack with fixed ΔP,

$$H\Delta P = 2n\bar{\tau}L \quad (4)$$

where $\bar{\tau}$ is the L-averaged shear stress on the board surface. In writing equation (4) we have assumed that the board thickness is small enough so that the force experienced by each board is dominated by skin friction over the L-long faces. This assumption is equivalent to writing that in an order of magnitude sense,

$$\tfrac{1}{2}\rho U_\infty^2 t \ll 2\bar{\tau}L \quad (5)$$

which, in view of the definition of average skin friction coefficient $C_f = \bar{\tau}/(\rho U_\infty^2/2)$, means that we are assuming

$$\frac{t}{L} \ll 2C_f. \quad (6)$$

Combining equation (4) with the C_f definition we obtain

$$U_\infty = \left(\frac{H\Delta P}{n\rho L C_f}\right)^{1/2}. \quad (7)$$

The total heat transfer rate through one board surface (i.e. across one boundary layer) is

$$q'_1 = \overline{q''}L = St L \rho c_P U_\infty (T_{max} - T_\infty) \quad (8)$$

where $\overline{q''}$ is the L-averaged heat flux. The Stanton number $St = \overline{q''}/\rho c_P U_\infty (T_{max} - T_\infty)$ can be evaluated by invoking the Colburn analogy between momentum and heat transfer in

NOMENCLATURE

c_P	specific heat of coolant [J kg^{-1} K^{-1}]
C_f	skin friction coefficient
D	plate-to-plate spacing [m]
D_h	hydraulic diameter, $2D$ [m]
f	friction factor
H	overall thickness of stack [m]
L	length of stack [m]
$\dot{m}'$	mass flowrate through stack [kg m^{-1} s^{-1}]
n	number of plates, $H/(D+t)$
Pr	Prandtl number
q'	stack-ambient heat transfer rate [W m^{-1}]
$\overline{q''}$	average heat flux [W m^{-2}]
Re_{D_h}	Reynolds number, UD_h/ν
Re_L	Reynolds number, U_0L/ν
St	Stanton number
t	plate thickness [m]
T_{max}	plate temperature level [K]
T_∞	coolant temperature [K]
U	mean velocity [m s^{-1}]
U_0	free stream velocity [m s^{-1}]
U_∞	core velocity [m s^{-1}].

Greek symbols

α	thermal diffusivity [m^2 s^{-1}]
ΔP	pressure difference [N m^{-2}]
μ	viscosity [kg s^{-1} m^{-1}]
ρ	density [kg m^{-3}]
$\bar{\tau}$	average wall shear stress [N m^{-2}].

Subscripts

$(\ )_{max}$	maximum
$(\ )_{opt}$	optimal.

turbulent boundary layer flow,

$$St = \tfrac{1}{2}C_f\, Pr^{-2/3} \quad (Pr \gtrsim 0.5). \tag{9}$$

In the end, for the total heat transfer rate removed from the stack we write $q' = 2nq'_1$, and obtain the following asymptotic expression for the overall conductance:

$$\left(\frac{q'}{T_{max}-T_\infty}\right)_{D\to\infty} = c_P H\, Pr^{-2/3}\left(\frac{\rho L C_f \Delta P}{t+D}\right)^{1/2}. \tag{10}$$

Intersection of the asymptotic regimes

Equations (3) and (10) show that the overall conductance increases with D when D is small, and decreases when D is large. This means that $q'/(T_{max}-T_\infty)$ is maximum at an optimal intermediate spacing that is of the same order of magnitude as the D value obtained by intersecting equations (3) and (10). The result of this intersection is given implicitly by

$$\frac{D_{opt}/L}{(1+t/D_{opt})^{1/2}} = (fC_f)^{1/2}\, Pr^{-2/3} \quad (Pr \gtrsim 0.5). \tag{11}$$

The corresponding maximum value of the overall thermal conductance is obtained by substituting $D = D_{opt}$ in equation (10) or equation (3):

$$\left[\frac{q'L}{kH(T_{max}-T_\infty)}\right]_{max} \lesssim \left(\frac{C_f}{f}\right)^{1/4} Pr^{1/6} \times\left(1+\frac{t}{D_{opt}}\right)^{-3/4}\left(\frac{\Delta P\cdot L^2}{\mu\alpha}\right)^{1/2} \quad (Pr \gtrsim 0.5). \tag{12}$$

The inequality sign is a reminder that if q' is plotted on the ordinate and D on the abscissa, the peak of the actual q' vs D curve is located under the intersection of the asymptotes (3) and (10). The right side of equation (12) represents the correct order of magnitude of the maximum overall thermal conductance, and can be expected to anticipate within 30% the exact value [10].

RESULTS AND CONCLUSIONS

Smooth surfaces

Beyond this point we must make certain assumptions regarding the values of the friction factor and skin-friction coefficient. If all the board surfaces are smooth, we can use the standard correlations [11]

$$f = 0.046 Re_{D_h}^{-1/5} \quad (10^4 < Re_{D_h} < 10^6) \tag{13}$$

$$\tfrac{1}{2}C_f = 0.037 Re_L^{-1/5} \quad (10^6 < Re_L < 10^8) \tag{14}$$

where $D_h = 2D$, $Re_{D_h} = 2DU/\nu$ and $Re_L = U_\infty L/\nu$. These allow us to relate U and U_∞ to ΔP, by combining equations (2) and (13) for U, and equations (7) and (14) for U_∞:

$$U = 5.98 D^{2/3}\nu^{-1/9}\left(\frac{\Delta P}{\rho L}\right)^{5/9} \tag{15}$$

$$U_\infty = 4.25 L^{-4/9}\nu^{-1/9}\left[\frac{\Delta P(D+t)}{\rho}\right]^{5/9}. \tag{16}$$

Combined, equations (13)–(16) express f and C_f as functions of the imposed pressure drop, i.e. functions that can be substituted on the right side of equation (11). The final expression for the optimal spacing is

$$\frac{D_{opt}/L}{(1+t/D_{opt})^{4/11}} = 0.071\, Pr^{-5/11}\left(\frac{\Delta P\cdot L^2}{\mu\alpha}\right)^{-1/11}. \tag{17}$$

The geometric meaning of this conclusion becomes clearer if we estimate the expected order of magnitude of the right side of equation (17). First, note that the Re_{D_h} range listed in equation (13) can be rewritten in terms of ΔP by using equation (15) and the assumptions that $(1+t/D_{opt})^{4/11} \cong 1$ and $Pr = 0.72$ (air):

$$0.09 > \left(\frac{\Delta P\cdot L^2}{\mu\alpha}\right)^{-1/11} > 0.032. \tag{18}$$

Similarly, the Re_L range specified in equation (14) can be rewritten using equation (16):

$$0.087 > \left(\frac{\Delta P\cdot L^2}{\mu\alpha}\right)^{-1/11} > 0.038. \tag{19}$$

Equations (18) and (19) show that the specified Re_{D_h} and Re_L ranges correspond to the same range of the pressure drop group $\Delta P\cdot L^2/\mu\alpha$. Taken together, equations (17)–(19) show that the slenderness ratio of each board-to-board channel (D_{opt}/L) takes values between approximately 0.003 and 0.007, and is relatively insensitive to changes in the applied pressure difference.

When the surfaces are smooth cf equations (13), (14), the maximum overall conductance expression (12) becomes

$$\left[\frac{q'L}{kH(T_{max}-T_\infty)}\right]_{max} \lesssim 0.57 Pr^{4/99} \times\left(1+\frac{t}{D_{opt}}\right)^{-67/99}\left(\frac{\Delta P\cdot L^2}{\mu\alpha}\right)^{47/99}. \tag{20}$$

In the case of a fluid with Prandtl number of order 1, equation (20) is almost the same as the more general equation (12) with the constant 0.57 in place of $(C_f/f)^{1/4}$. In conclusion, the maximum overall conductance increases almost as $\Delta P^{1/2}$.

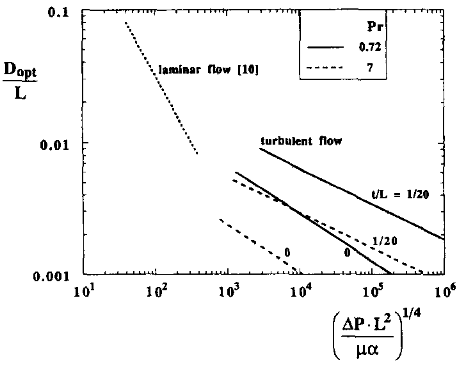

FIG. 2. The optimal spacing as a function of the pressure difference number, the Prandtl number and the plate slenderness ratio.

Figure 2 shows the optimal spacing calculated by using equation (17) for turbulent flow. The corresponding D_{opt}/L result for laminar flow [10] has been plotted to the left,

$$\frac{D_{opt}}{L} \cong 3.2\left(\frac{\Delta P \cdot L^2}{\mu\alpha}\right)^{-1/4}. \tag{21}$$

The figure shows that when the flow is turbulent D_{opt}/L depends not only on the pressure difference number $(\Delta P \cdot L^2/\mu\alpha)$ but also on Pr and t/L. The optimal spacing in turbulent flow increases as Pr and t/L increase, and is quite sensitive to such changes.

Stack immersed in a free stream

Another way of interpreting the information of equation (17) and Fig. 2 is to consider a cooling arrangement in which specified is not ΔP but the coolant velocity well upstream of the stack, U_0. In such an arrangement, the scale of ΔP across each channel is of the order of [12]

$$\Delta P \cong \tfrac{1}{2}\rho U_0^2, \tag{22}$$

which can be substituted in equations (17) and (21) to obtain

$$\frac{D_{opt}/L}{(1+t/D_{opt})^{4/11}} \cong 0.076 Pr^{-6/11}\, Re_L^{-2/11} \quad \text{(turbulent)} \tag{23}$$

$$\frac{D_{opt}}{L} \cong 3.8 Pr^{-1/4}\, Re_L^{-1/2} \quad \text{(laminar)}. \tag{24}$$

The Reynolds number Re_L is based on the specified upstream velocity and the flow length of the stack,

$$Re_L = \frac{U_0 L}{\nu}. \tag{25}$$

The optimal spacings recommended by equations (23), (24) are displayed in Fig. 3, which shows that in turbulent flow the channel spacing is influenced not only by Re_L but also by Pr and t/L.

The maximum overall thermal conductance (20) can also be expressed in terms of Re_L by using equation (22):

$$\left[\frac{q'L}{kH(T_{max}-T_\infty)}\right]_{max} \leqslant 0.41 Pr^{51/99}\left(1+\frac{t}{D_{opt}}\right)^{-67/99} Re_L^{94/99}. \tag{26}$$

The t/D_{opt} ratio appearing on the right side is given by equation (23). The resulting maximum thermal conductance estimate has been plotted in the lower frame of Fig. 4, next to the corresponding curves known for laminar flow. The upper frame of Fig. 4 shows the same results by using the pressure drop number on the abscissa. The turbulent flow curves were obtained by combining equations (20) and (17).

The interesting conclusion made visible by the two frames of Fig. 4 is that the turbulent portion of each curve is, in an order of magnitude sense, an extension of the laminar

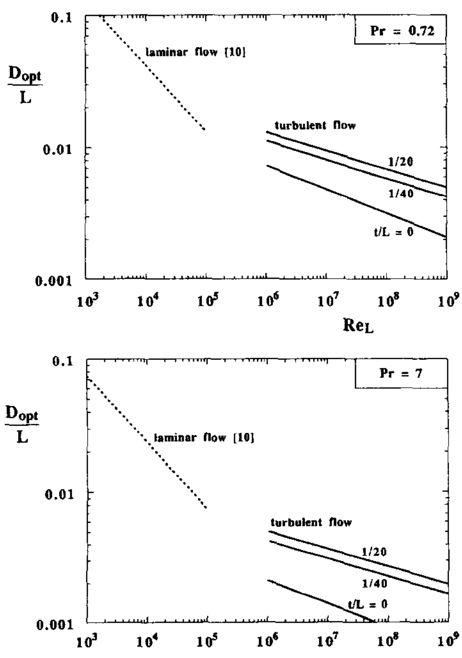

FIG. 3. The optimal spacing as a function of $Re_L = U_0L/\nu$, the Prandtl number and the plate slenderness ratio.

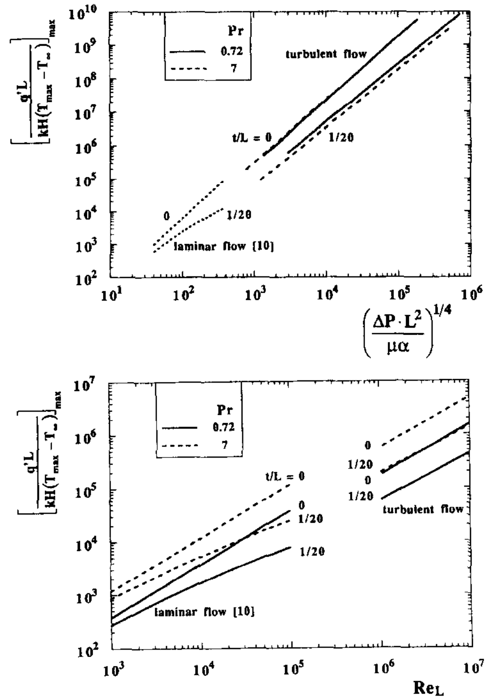

FIG. 4. The maximum overall stack-ambient thermal conductance as a function of the imposed pressure difference (top), or the free-stream velocity (bottom).

portion. This feature is unlike in Figs. 2 and 3, where there is a definite change in the behavior of D_{opt}/L as the flow regime changes.

Acknowledgement—This work was sponsored by the IBM Corporation, Research Triangle Park, NC, under the guidance of Dr Sang W. Lee.

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Pergamon

Int. J. Heat Mass Transfer. Vol. 37, No. 6, pp. 1048–1050, 1994

0017–9310/94 $6.00+0.00

0017–9310(93)E0021–8

Temperature distribution within vortices in the wake of a cylinder

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(*Received* 15 *June* 1993 *and in final form* 22 *October* 1993)

INTRODUCTION

The transport equations for the vorticity ω and temperature T of a line vortex which is diffusing into the surrounding (ambient temperature) fluid are

$$\frac{\partial \omega}{\partial t} = \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \omega}{\partial r}\right) \quad (1)$$

and

$$\frac{\partial T}{\partial t} = \frac{\alpha}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right). \quad (2)$$

At $t = 0$, the circulation Γ_0 and thermal energy Q_0 are concentrated along the axis of rotation. Solutions to (1) and (2) are

$$\omega = \frac{\Gamma_0}{4\pi\nu t}\exp\left(-\frac{r^2}{4\nu t}\right) \quad (3)$$

and

$$T = \frac{Q_0}{4\pi\rho c_p \alpha t}\exp\left(-\frac{r^2}{4\alpha t}\right), \quad (4)$$

respectively. Equation (3) is given in a number of texts, e.g. refs. [1, 2]. Equation (4) was given in ref. [3] in the context of a line source of heat instantaneously released into an infinite solid. At the vortex centre, the vorticity and temperature are (at time t)

$$\omega_c = \frac{\Gamma_0}{4\pi\nu t} \quad (5)$$

and

$$T_c = \frac{Q_0}{4\pi\rho c_p \alpha t}. \quad (6)$$

The distributions for ω and T can be re-written in normalised form

$$\frac{\omega}{\omega_c} = \exp\left[-0.693 Pr^{-1}\left(\frac{r}{R}\right)^2\right] \quad (7)$$

and

$$\frac{T}{T_c} = \exp\left[-0.693\left(\frac{r}{R}\right)^2\right], \quad (8)$$

where the half-radius R is given by

$$\frac{R^2}{4\alpha t} = 0.693. \quad (9)$$

The vorticity distribution for vortices in the laminar wake ($Re_d = 140$) behind a cylinder was indirectly measured by Okude and Matsui [4] and was found to be in reasonable agreement with equation (7). To our knowledge, equation (8) has not been verified experimentally. This is surprising